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THERE IS NO EXACTLY k-TO-1 FUNCTION FROM ANY CONTINUUM ONTO $[0,1]$, OR ANY DENDRITE, WITH ONLY FINITELY MANY DISCONTINUITIES

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ABSTRACT. Katsuura and Kellum recently proved [8] that any (exactly) k-to-1 function from $[0,1]$ onto $[0,1]$ must have infinitely many discontinuities, and they asked if the theorem remains true if the domain is any (compact metric) continuum. The result in this paper, that any (exactly) k-to-1 function from a continuum onto any dendrite has finitely many discontinuities, answers their question in the affirmative.

1. Introduction. Continuous (exactly) k-to-1 maps have been extensively studied for decades. Much research has concentrated on which spaces can be the domain of such a map, and for which k (see bibliography). As for which spaces can be the image of such a map, Harrold [5] showed that no arc could be (for any $k > 1$), and he showed that if the domain is a simple graph, then the image contains a simple closed curve. Recently Nadler and Ward [12] proved that if the image Y is locally connected, then there is a k-to-1 map onto Y iff Y contains a simple closed curve. They also proved that any continuum (locally connected or not) that contains a nonunicoherent subcontinuum is the image of a k-to-1 map.

If a discontinuity or two is allowed for the k-to-1 function, more spaces qualify for both domain and range, not surprisingly. For instance, K. Kuperberg [9] has constructed a 2-to-1 function on a disk with one discontinuity, and Kellum and Katsuura [8] showed that for k odd or $k = 4$, there is a k-to-1 function from $[0,1]$ into $[0,1]$ with exactly one discontinuity. The author has shown in [7] that if k is even and $k > 4$ then there is a k-to-1 function from $[0,1]$ into $[0,1]$ with two discontinuities (and none with fewer than two), and has shown in [6] that every 2-to-1 function from $[0,1]$ to any Hausdorff space has infinitely many discontinuities. Kellum and Katsuura also showed that if the image is compact, then any function from $[0,1]$ to $[0,1]$ requires infinitely many discontinuities for $k > 1$. In the same paper [8], Kellum and Katsuura ask if every k-to-1 function from any continuum onto $[0,1]$ must have infinitely many discontinuities. The main result of this paper is that any (exactly) k-to-1 function from any continuum onto a locally connected continuum with no simple closed curve (a dendrite) has infinitely many discontinuities. This answers the Katsuura-Kellum question in the affirmative.

Requiring no simple closed curve in the image is clearly necessary since otherwise there is a continuous map [12]. In view of Nadler and Ward's similar result with nonunicoherent subcontinua, the following seems a natural question:

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Question. Is there a k-to-1 function from any continuum onto any arc-connected, hereditarily unicoherent continuum with only finitely many discontinuties?

In the Katsuura-Kellum result mentioned above, compactness of the image is crucial, and it is for this paper's result also, as the following simple example demonstrates:

EXAMPLE. Let X be the dendrite in the plane that is the union of straight line segments with one endpoint at the origin $(0,0)$ of length $1/i$ and slope i for each positive integer i, and let $Y = [0,1)$. Define $f((0,0)) = 0$ and divide the countably many half open, half closed intervals left in $X - \{(0,0)\}$ into k disjoint infinite collections, $G_1, G_2, \ldots, G_k$. For each $i < k$, map the first arc in G_i homeomorphically to $[0,1/2)$, the second to $[1/2,3/4)$, etc. Map the first arc in the last collection G_k to $(0,1/2]$, the second to $(1/2,3/4]$, etc. Then f is k-to-1 and is discontinuous only at $(0,0)$.

2. Some definitions. A *continuum* is a compact connected metric space.

A function is *k-to-*1 if each point in the image has exactly k inverse points.

A *dendrite* is a locally connected continuum containing no simple closed curve.

A sequence of sets is *null* if their diameters converge to 0.

The arc A is *irreducible from p to S* if one endpoint of A is the point p, the other endpoint of A lies in the set S, and no point of A other than the two endpoints lies in S.

3. Preliminary lemmas. Some version of Lemma 0 was independently noticed by Kellum and Katsuura.

LEMMA 0. *If f is a finite-to-one function from the compactum X to the dendrite Y with only finitely many discontinuities, then X does not contain an infinite collection of disjoint continua whose diameters are bounded away from* 0.

PROOF. Suppose there is such a collection of disjoint continua. Then there is a subsequence $\{C_i\}$ that converges to the nondegenerate continuum C'. Since there are only finitely many discontinuities, C' contains a nondegenerate continuum C that contains no discontinuity for f, and since all of C cannot map to the same point, there are points r and s in C that map to different points in Y. Let t denote an interior point of the arc from $f(r)$ to $f(s)$ in Y; then t is a cut point of Y and there are open sets R and S containing $f(r)$ and $f(s)$ so that any continuum in Y that intersects both R and S also contains t.

If i is large enough then C_i contains no discontinuity of f and contains points close enough to r and s that $f(C_i)$ is a continuum in Y that intersects both R and S, and hence contains t. This makes infinitely many point inverses for t since the C_i are disjoint, a contradiction.

COROLLARY 1 TO LEMMA 0. *If f is a finite-to-*1 *function from the continuum X to the dendrite Y with only finitely many discontinuities, then X is locally connected and hence arc-wise connected.*

COROLLARY 2 TO LEMMA 0. *The statement of Lemma* 0 *remains true if Y is a continuum with the property that each pair of points in Y is separated by some finite subset of Y.*

COROLLARY 3 TO LEMMA 0. *If f is a finite-to-1 function from a compactum to a dendrite Y with finitely many discontinuities, and if $[a,b]$ is an arc in X, then $\lim\{f(x)|x \in [a,b], x \to b\}$ exists.*

PROOF. Otherwise there is a sequence of disjoint subarcs in $[a,b]$ converging to b whose left endpoints map to a sequence in Y that converges to the point s in Y and whose right endpoints' images converge to a different point r in Y. As in the proof of Lemma 0, the images of the disjoint subarcs map arbitrarily close to both r and s and hence most of the arcs map to a point t in the arc from r to s, yielding the same contradiction. Q.E.D.

Lemma 1 essentially reduces the theorem to the case where Y is an arc although that may not be apparent yet. It has as a corollary that if all of the discontinuities of f map to two points in Y and A is the arc between them, then $f^{-1}(A)$ is a continuum so that $f \upharpoonright f^{-1}(A)$ has all of f's properties and maps onto an arc. (To see that $f^{-1}(A)$ is connected, let a and b be any two points of X that map to A and note that an arc in X from a to b maps to A by Lemma 1 so that a and b are arc-connected in $f^{-1}(A)$.) The fact that the discontinuities are probably more widely dispersed will cause some difficulties, but there will still be an arc A in Y where $f \upharpoonright f^{-1}(A)$ yields a contradiction.

LEMMA 1. *Suppose that f is a k-to-1 function from the continuum X onto the dendrite Y whose set of discontinuities,* DIS, *is finite, that E' is a 1-complex in Y containing f*(DIS), *and that $[x,y]$ is an arc in X with $f(x)$ and $f(y)$ both in E'. Then $f([x,y])$ is a subset of E'.*

PROOF. Let E denote $f^{-1}(E')$. Since DIS $\subset E$ and E' is compact, E is closed in X and contains both x and y by assumption. Hence $[x,y] - E$ has an open component (a,b), unless $f([x,y]) \subset E'$ as desired. Since (a,b) is connected and misses DIS, its image $f((a,b))$ is connected and lies in some component C of $Y - E'$. Let A be an irreducible arc from some point of $f((a,b))$ to E', which connects $f((a,b))$ to E' on the off chance that a and b are both in DIS and $\mathrm{Cl}(f((a,b)))$ is in C. If $\mathrm{Cl}(f((a,b)))$ intersects E', let $A = \emptyset$.

Since every subcontinuum of a dendrite is a dendrite, $D = A \cup \mathrm{Cl}(f((a,b)))$ is a dendrite with one endpoint e in E' and another q_0 not in E'. Every dendrite has at least two endpoints and if D had two in E' then there would be two arcs between them in Y, one in D and one in E', not possible since dendrites are uniquely arcwise connected.

The initial dendrite D will be augmented with images of special arcs (defined below as "eligible") getting larger dendrites at each ordinal stage because f is k-to-1. The contradiction is that the process does not end and also cannot become uncountable.

DEFINITION. The arc $[w,z]$ in X is *eligible from* w if $[w,z)$ misses E and z is in E.

DEFINITION. The eligible arc $[w,z]$ from w *extends* the dendrite D' through its endpoint q if $f(w) = q$, and for some subarc $[w,x']$ of $[w,z)$, q separates $f([w,x']) - \{q\}$ from e in Y. The dendrite $D' \cup \mathrm{Cl}(f([w,z)))$ is the *extension*.

(All of the dendrites that will be considered will contain the special endpoint e in E' described before.)

We will extend the dendrite $D = A \cup \mathrm{Cl}(f((a,b)))$ first. Recall that q_0 is an endpoint of D not in E' and that e is the endpoint of D in E'. Denote by B the arc from q_0 to e in E'. Let $w_1, w_2, \ldots, w_k$ be the k points in $X - E$ that map to q_0, and let $U_1, U_2, \ldots, U_k$ be disjoint open sets in $X - E$ containing them that map into C with the further property that if w_i is not in $[a,b]$ then U_i also misses $[a,b]$. Since X is arc-connected by Lemma 0 there is for each i an eligible arc R_i from w_i and a subarc R'_i in U_i with endpoint w_i.

We wish to show that one of the R_i arcs extends D through q_0, where R'_i satisfies the separation property. So suppose that for each i there is a point x_i in R'_i such that the arc T_i from $f(x_i)$ to e does not contain q_0. Since $f(R'_i)$ contains an arc from q_0 to $f(x_i)$, T_i connects $f(x_i)$ to e, and B is an arc from q_0 to e, $B \subset f(R'_i) \cup T_i$. Since q_0 is not in T_i, some initial segment $B_i = [q_0, q_i]$ of B is contained in $f(R'_i)$. Let $N = \bigcap\{B_i | i = 1, 2, \ldots, k\}$, say $N = [q_0, q']$. Each point of N has a point inverse in R'_i and there are k disjoint R'_i arcs, so all of the point inverses of N are accounted for in the union of the R'_i arcs.

At this point the argument for the initial dendrite differs from that of later dendrites. If w_i belongs to the arc (a,b) then both $[w_i, a]$ and $[w_i, b]$, subarcs of $[a,b]$, are eligible from w_i since a and b are in E and (a,b) misses E. One of them, say $[w_i, a]$, satisfies $f([w_i, a']) \cap N = \{q_0\}$ for some a' in $(w_i, a]$, since each point of N has only one point inverse in each U_i and since $[w_i, a'] \cap [w_i, b'] = \{w_i\}$. But this means that $[w_i, a]$ extends D through q_0. So we may assume that no w_i is in (a,b).

Since each point inverse of q_0 misses (a,b), q_0 is not in $f((a,b))$, so $q_0 = \lim\{f(x) | x \in (a,b),\ x \to a\}$, or b of course, from the Corollary to Lemma 0. No w_i is a or b since a and b are in E so each w_i is not in $[a,b]$ and the open set U_i containing w_i misses $[a,b]$. The irreducible arc A, if nonempty, has one endpoint e in E' and the other endpoint in $f((a,b))$ which cannot be an endpoint of the union $D = A \cup \mathrm{Cl}(f((a,b)))$, so q_0 is not in A. This means that the arc B from q_0 to e has an initial segment in $f((a,b)) \cup \{q_0\}$, so N must intersect $f((a,b))$. This is a contradiction since each point inverse of N is in some U_i and every U_i misses $[a,b]$.

Therefore for some least i, q_0 separates $f(R'_i) - \{q_0\}$ from e. Let e' denote the endpoint of R_i in E. Define $D_1 = D \cup \mathrm{Cl}(f(R_i - \{e'\}))$. Note that the larger R_i is used, and note that $\mathrm{Cl}(f(R_i)) - f(R_i)$ is a single point, or is empty, from the Corollary to Lemma 0.

Now suppose that D_α has been constructed for each $\alpha < \beta$ so that:

(1) D_α is a dendrite in $C \cup \{e\}$ containing D,

(2) if q is any endpoint of D_α other than e, then there is an eligible arc in X that extends D_α through q,

(3) if α is not a limit ordinal then there is an eligible arc $[w,z]$ that extends $D_{\alpha-1}$ through some endpoint of $D_{\alpha-1}$ and $D_\alpha = D_{\alpha-1} \cup \mathrm{Cl}(f([w,z)))$, and

(4) if α is a limit ordinal, then $D_\alpha = \mathrm{Cl}(\bigcup\{D_\gamma | \gamma < a\})$.

So far $D_0 = D$ satisfies (1), (2), and (3), and D_1 satisfies (1) and (3). We will construct D_β.

First, assume that $\beta - 2$ exists. There is an eligible arc $[w,z]$ that extends $D_{\beta-2}$, and $D_{\beta-1} = D_{\beta-2} \cup \mathrm{Cl}(f([w,z)))$. The dendrite $D_{\beta-1}$ has an endpoint q in C since it cannot have two endpoints in E' (lest Y have a simple closed curve). Choose the endpoint q in the new part $\mathrm{Cl}(f([w,z)))$. As before there are k points $w_1, \ldots, w_k$ in $X - E$ that map to q, disjoint open sets $U_1, \ldots, U_k$ containing them that map into C and whose closures are in $X - E$. For each i there is an eligible arc from w_i, R_i, and a subarc R_i' in U_i with endpoint w_i. Again, if, for each i, q does not separate $f(R_i') - \{q\}$ from e, then there is an arc $N = [q, q']$ in the arc from q to e such that each point of N has a point inverse in each R_i'.

Now suppose that some w_i is in $[w, z)$. Since the new endpoint q is not the old, $f(w)$, w_i is in (w, z). But $f([w, w_i])$ contains an arc from the old endpoint to q and must then map onto N. This means the eligible arc $[w_i, z]$ has no point in U_i other than w_i that maps to N, so $[w_i, z]$ extends $D_{\beta-1}$. Hence we will assume that no w_i is in $[w, z)$.

Thus q is not in $f([w, z))$, so $q = \lim\{f(x) | x \in [w, z), x \to z\}$. But this means there are points of $[w, z)$ arbitrarily close to E that map to N, contradicting the fact that each $\mathrm{Cl}(U_i)$ misses E and the U_i have all the point inverses of N.

Thus, for some least i $[w_i, z_i]$ extends $D_{\beta-1}$ through q; define $D_\beta = D_{\beta-1} \cup \mathrm{Cl}(f([w_i, z_i]))$. Then properties (1) and (3) hold for D_β. We proved that the new endpoints of $D_{\beta-1}$ extend, and the old ones do by induction. Hence $D_{\beta-1}$ also satisfies (2).

Now suppose that $b - 2$ does not exist, that is that $\beta - 1$ is a limit ordinal and $D_{\beta-1} = \mathrm{Cl}(\bigcup\{D_\gamma | \gamma < \beta - 1\})$. If the endpoint q of $D_{\beta-1}$ is in some D_γ for $\gamma < \beta - 1$ then some eligible arc extends $D_{\beta-1}$ through q by the induction hypothesis. So assume q is an endpoint of $D_{\beta-1}$ that is not in any previous D_γ. Exactly as in the nonlimit case, there is an arc $N = [q, q']$ in the arc from q to e and k disjoint open sets $U_1, \dots, U_k$ whose closures miss E such that $f^{-1}(N)$ lies in the union of the U_i open sets.

Each subarc $[q, q'']$ of N has a point $f(x)$ that is from some previous extension, i.e. x belongs to some eligible arc that extended some D_γ for $\gamma < \beta - 1$. Otherwise, the arc $[q, q'']$ is in $\lim_{\beta-1} = D_{\beta-1} - \bigcup\{D_\gamma | \gamma < \beta - 1\}$. Since Y is locally connected there is an ordinal $\gamma 1$ with points of $D_{\gamma 1}$ close enough to q and q'' that there are small disjoint arcs connecting each of q and q'' to $D_{\gamma 1}$. But $[q, q'']$ in $\lim_{\beta-1}$ misses the arc in $D_{\gamma 1}$ connecting the small arcs, and a simple closed curve is formed in Y. Hence $\lim_{\beta-1}$ contains no arc.

Hence there is a sequence of points $\{f(x_i)\}$ in N converging to q such that each x_i is in an eligible arc. Since q belongs to no previous D_γ, the eligible arcs can come from different extensions and hence are disjoint. But each contains a point of E. Since the sequence is null, it converges to some point of E, and eventually the x_i are outside of the $\mathrm{Cl}(U_i)$ and map to N; a contradiction.

Thus all of the endpoints of $D_{\beta-1}$ extend, and D_β can be defined as $D_{\beta-1}$ plus one of the extensions.

This completes the induction and Lemma 1 is proved.

The following real analysis lemmas will be needed later:

LEMMA 2. *Suppose that f is a continuous map from $(0,1)$ to $[0,1]$ that is at most k-to-1. Then there is a subinterval of $(0,1)$ on which f is 1-to-1.*

PROOF. Let c be one of the values in $[0,1]$ with the maximum number, j, of point inverses, $j \leq k$. If $a_1, a_2, \dots, a_j$ are the points that map to c subscripted so that $a_i < a_{i+1}$, then the graph between a_i and a_{i+1} is either a hill (i.e. $f(x) > c$ for x between a_i and a_{i+1}), or a valley. Let m be the number of hills and n the number of valleys. Then $m+n=j-1$. Suppose $m \geq n$. Let

$$L = \min_i \{\text{lub}\{f(x) | a_i < x < a_{i+1}\}\}.$$

Suppose the graph over (a_1, a_2) is a hill. If f is not 1-to-1 over (a_1, b_1) where b_1 is the least point of (a_1, a_2) that maps to L, then some horizontal line $\{y=r\}$ with $c < r < L$ intersects the graph of f at least three times between a_1 and b_1. This same line intersects the graph between a_1 and a_j at least $2m+2$ times. But $2m+2 \geq m+n+2 = j+1 > j$, contradicting the maximality of j.

LEMMA 3. *If f is a finite-to-1 continuous map from $[0,1)$ onto $[0,1)$, then there is a collection C of disjoint half open, half closed arcs in the domain $[0,1)$ such that f restricted to $[0,1) - \bigcup C$ is 1-to-1 and still onto $[0,1)$.*

PROOF. Let x_1 be the largest number in $[0,1)$ that maps to 0. If x_1 is not 0, let I_1, the first member of C, be $[0, x_1)$ and let f_1 be the restriction of f to $[x_1, 1)$. If $x_1 = 0$, set $f_1 = f$. Well order the rational numbers in $(0,1)$, $r_1, r_2, \ldots$. If $f_1^{-1}(r_1)$ has more than one point, let s_1 denote the least and t_1 the greatest. Put $I_2 = [s_1, t_1)$ in C and denote by f_2 the restriction of f_1 to $[x_1, s_1) \cup [t_1, 1)$. If $f^{-1}(r_1)$ has only one point, let $f_2 = f_1$.

Note that since $f([0,1))$ has no maximum, $\lim\{f(x) | x \to 1\} = 1$. Hence $f_2^{-1}(r_2)$ either lies in $[0, s_1)$ or it lies in $[t_1, 1)$. If more than one point in the domain of f_2 maps to r_2, let s_2 denote the least and t_2 the greatest. Again add $I_3 = [s_2, t_2)$ to C and note that the three elements of C are disjoint. Call f_3 the restriction of f_2 to its domain with I_3 removed. If only one point maps to r_2 via f_2 let $f_3 = f_2$.

In this way, maps $f_1, f_2, \ldots$ are constructed so that:

(1) $f_j^{-1}(r_i)$ has only one point for $i < j$,

(2) the image of f_j is $[0,1)$,

(3) f_j is either f_{j-1} or there is a half open, half closed interval I_j in the domain D of f_{j-1} so that f_j is f_{j-1} restricted to $D - I_j$, and

(4) if y is any number in $[0,1)$ then every x in the domain of f_j that maps to y lies in the same component of the domain of f_j.

Let g denote the intersection of $\{f_i | i = 1, 2, \ldots\}$, and let y be any number in the image $[0,1)$. We will show that there is exactly one x in the domain of g that maps to y.

First, there is at least one x in the domain of g that maps to y. For each of the finitely many x in $[0,1)$ that maps to y (via f), let i_x be the least positive integer such that x is not in the domain of f_{ix} (if there is one). Let j be the largest of these i's. Then since f_j is onto, some x has no i_x and this x will be in the domain of g.

Second, there cannot be two numbers, say x and w, in the domain of g that map to y. If so, then from prperty (4) above, the segment $[x, w]$ must be in a component of the domain of every f_i since x and w are. But f cannot be constant on $[x, w]$ so there is some rational number r_j such that two numbers between x and w map to r_j. Since those two numbers are not both in the domain of f_{j+1}, the interval $[x, w]$ is not in the domain of f_{j+1} after all. A contradiction.

4. Theorem and proof.

THEOREM. *If f is a k-to-1 function from the continuum X onto the dendrite Y and $k > 1$, then f must have infinitely many discontinuities.*

PROOF. Suppose on the contrary that the set DIS of discontinuities of f is finite. Let E' be a minimal 1-complex containing $f(\text{DIS})$ and let J be $f(\text{DIS})$ plus the junction points of E'. J is finite with, say, n elements, so $E' - J$ has $n-1$ open arcs, $A_1, A_2, \ldots, A_{n-1}$. Let $D = f^{-1}(J)$; then D has kn points.

From Lemma 0, X is arc-connected, so there is an arc, B_1, between two points of D that otherwise misses D, and a second irredubible arc B_2 from a third point of D to B_1, etc., getting a 1-complex E such that: $E - D = \bigcup\{\text{Int}(B_i) | i = 1, 2, \ldots, kn-1\}$. Since D maps to E', it follows from Lemma 1 that all of E maps to E'.

Since $D = f^{-1}(J)$ and each $\text{Int}(B_i)$ misses D and is connected, each $\text{Int}(B_i)$ maps into some A_j. There are $kn-1$ B's, $n-1$ A's, and $k > 1$, so at least $k+1$ of the B's, say $B_1, B_2, \ldots, B_{k+1}$, map to the same A, say, A_1.

We will be primarily interested in those points of X that map to A_1. If p maps to A_1 and T is an arc from p to E, then $\text{Int}(T)$ maps to A_1 too, by Lemma 1 again. Let $X' = D \cup f^{-1}(A_1)$. Then:

(1) f is k-to-1 on X' to $J \cup A_1$,

. (2) X' is compact, and

(3) if p is in $X' - D$, there is an arc in X' from p to D.

MAIN CLAIM (proof later). $X' - (D \cup B_1 \cup \cdots \cup B_{k+1})$ is the union of disjoint open arcs and half open, half closed arcs.

Let M denote the collection of arcs from the main claim plus the interiors of the B_i arcs, $i = 1, 2, \ldots, k+1$. Then M decomposes $X' - D$. For each open arc in M there is an open sub arc on which f is 1-to-1 (Lemma 2). Change the composition of M by replacing each of its open arcs with an open arc on which f is 1-to-1 plus the 0, 1 or 2 leftover half open, half closed arcs. M still covers $X' - D$ and has at least $k+1$ open arcs, which will be denoted $R_1, R_2, \ldots, R_{k+1}$, with the subscripts arranged so that if A_1 is identified with $[0,1]$, then $\text{glb}\, f(R_i) \leq \text{glb}\, f(R_{i+1})$ for $i = 1, 2, \ldots, k$. For bookkeeping purposes, color the open arcs in M blue and the half open, half closed arcs black.

For each $i = 1, 2, \ldots, k$, disjoint subsets G_i of $X' - D$ will be constructed so that f maps G_i onto $(\text{glb}\, f(R_i), 1)$ in A_1, and so that G_i misses R_{k+1}. The contradiction will be that each point of $f(R_{k+1})$ has one inverse in R_{k+1} and k others in the G's, one too many for a k-to-1 function.

Construction of the G_i. Each G_i will contain R_i and black points from the half open, half closed arcs of M, and when G_i is constructed its points will be colored green to distinguish them from the other points in M. Since the construction of G_1 is similar to any other except that there are no green points yet, we will start with the inductive step:

Suppose G_i has been constructed for each $i < j$, so that:

(1) $f(G_i) = (\text{glb}\, f(R_i), 1)$,

(2) f is 1-to-1 on G_i,

(3) G_i is a subset of the union of the black arcs in M plus R_i,

(4) $G_i \cap G_n = \varnothing$ if $i \neq n$, and

(5) the collection M is altered but still has R_n for $n \geq j$, covers the portion of $X' - D$ not in $\bigcup\{G_i | i < j\}$, and still consists of blue open arcs on which f is 1-to-1 and black half open, half closed arcs.

Let $y = \operatorname{lub} f(R_j)$. If $y = 1$, let $G_j = R_j$ and color it green. Otherwise, suppose there are m blue points that map to y as well as the $j-1$ green points that map to y. Each of the blue points is in an open interval on which f is 1-to-1, so there is a positive number e such that there are at least $m+1$ blue points that map to $y-e$, including the one in R_j. The e can also be made small enough that there are still $j-1$ green points that map to $y-e$. Since $f^{-1}(y-e)$ has less than $k+1$ points, $(j-1)+(m+1) < k+1$. Hence there is at least one black point that maps to y.

The function f is not 1-to-1 on the black half open, half closed arcs, but the graph of f restricted to a black arc is at each of its points either a crossing point, a local maximum, or a local minimum. This is true of any finite-to-1 map, and f is continuous on each black, or blue, arc in M. For each black point that maps to y that is either a crossing or a local maximum, there is added to $f^{-1}(y-e')$, for some $e' < e$, at least one black point. Hence there is at least one black point, x, that maps to y and is a local minimum.

Suppose the half open, half closed arc in M that x belongs to is $[a, b)$. Either there is a first point c in $[a, b)$ between x and b at which $f \restriction [x, b)$ is a maximum, or $\operatorname{lub} f([x, b)) = \lim\{f(c) | c \to b\}$ in $[x, b)$. In the former case put $T_1 = [x, c)$ in H_j, a precursor of G_j, and in the latter case put $T_1 = [x, b)$ in H_j. The arc $[a, b) - T_1$ is the union of 0, 1, or 2 half open, half closed arcs; put them back in M and remove $[a, b)$ from M. Note that f maps $T_1 = [x, x')$ to $[y, \operatorname{lub} f(T_1))$, with $f(x) = y$.

Now suppose the half open, half closed arcs $T_1, T_2, \ldots, T_\alpha, \ldots$ have been constructed for all $\alpha < \beta$ so that:

(1) if $T_\alpha = [x_\alpha, x'_\alpha)$, then f satisfies the hypothesis of Lemma 3, i.e. $f(T_\alpha) = [f(x_\alpha), \operatorname{lub}(f(T_\alpha)))$,

(2) each T_α is black,

(3) if $f(T_\alpha) = [y_0, y_1)$ in $[0, 1)$, then there is a number $y_2 > y_1$ such that $f(T_{\alpha+1}) = [y_1, y_2)$, and

(4) if α is a limit ordinal, then there is a number y greater than

$$y_0 = \operatorname{lub}\{f(T_\gamma) | \gamma < \alpha\}$$

such that $f(T_\alpha)$ is $[y_0, y)$.

Now, if $y = \operatorname{lub}\{f(T_\alpha) | \alpha < \beta\}$ is 1, we are through constructing H_j. Otherwise, exactly as in the construction of T_2, T_β is formed. The only minor difference is that $f^{-1}(y-e)$ has at least one black point from $\bigcup\{T_\alpha | \alpha < \beta\}$, rather than a blue point from R_j. Continue until $y = 1$ is reached, finishing the construction of H_j.

Each $T_\alpha = [x_\alpha, x'_\alpha)$, for all relevant α, maps onto some $[y_\alpha, y_{\alpha+1})$ in $[0, 1]$ with $f(x_\alpha) = y_\alpha$ and $y_{\alpha+1} = \operatorname{lub} f(T_\alpha)$. From Lemma 3, there is a collection C_α of disjoint half open, half closed arcs in T_α such that $f \restriction (T_\alpha - \bigcup C_\alpha)$ is 1-to-1 and has the same $[y_\alpha, y_{\alpha+1})$ image. Color each point of $T_\alpha - \bigcup C_\alpha$ green and put it in the set G_j under construction. The half open half closed arcs of C_α stay black and are returned to the ever-changing collection M.

By induction $G_1, G_2, \ldots, G_k$ are defined and the theorem is proved.

PROOF OF MAIN CLAIM. All sets in this proof are presumed to be in X'. The property that each point in X' connects to D via an arc in X' is heavily used, as are the conclusions of Lemma 0 and Lemma 1, which are true for compact subsets of X.

Let $F = B_1 \cup B_2 \cup \cdots \cup B_{k+1} \cup D \subset X'$. We will find a collection C of disjoint open or half open, half closed arcs whose union is $X' - F$. If points were allowed in C, it would be easy. The secret is to cover the most complex arcs in $X' - F$ first. Complexity centers on the following definition:

The point p is an *offshoot* limit point of the set Q if there is an arc A containing p and a null sequence of arcs converging to p, each with one endpoint on A and one endpoint in $Q - A$.

The complexity levels are defined as follows: If x is in $X' - F$, then

(1) level$(x) \geq 0$,

(2) level$(x) \geq i$ if x is an offshoot limit point of points whose level is at least $i - 1$, and

(3) level$(x) = i$ if level$(x) \geq i$ but not level$(x) \geq i + 1$.

Many continua have arbitrarily high levels, but not one on which there is an at most k-to-1 function to $(0, 1)$:

Fact 1. No point in $X' - F$ has level $k + 1$ or higher.

PROOF. Suppose on the contrary that there is a point p_1 in $X' - F$ with level$(p_1) \geq k+1$. By definition there is an arc S_1 containing p_1 and a null sequence of offshoot arcs converging to p_1 with non-S_1 endpoints in level k or higher. Since p_1 is not in the closed set F, we may assume that S_1 and all of its offshoot arcs also miss F. Let $f(S_1) = [a_1, b_1]$ in $A_1 = (0, 1)$.

Now suppose for each $i < j < k + 2$ an arc S_i in $X' - F$ has been chosen and also a point p_j in $X' - F$ so that:

(1) $S_n \cap S_m = \varnothing$ if $n \neq m$ and both are less than j,

(2) $f(S_m) = [a_m, b_m] \subset (a_{m-1}, b_{m-1})$ for $1 < m < j$,

(3) p_j is not in any S_i, $i < j$, and

(4) p_j has level at least $k + 2 - j$ and maps into (a_{j-1}, b_{j-1}).

Note that each point in (a_{j-1}, b_{j-1}) has an inverse in each of the disjoint S_i, $i < j$.

By the definition of $(k + 2 - j)$-level, there is an arc S_j in $X' - F$ containing p_j with an attached null sequence of offshoot arcs converging to p_j with endpoints of level at least $k + 1 - j$. This S_j can be made short enough to miss the other S_i, $i < j$, since p_j misses them. Since p_j maps into (a_{j-1}, b_{j-1}) the arc S_j can also be made short enough that $f(S_j) = [a_j, b_j]$ lies in (a_{j-1}, b_{j-1}). Thus properties (1) and (2) are still true.

Since at most a finite number of disjoint arcs can map to any interval that contains either a_j or b_j and since each offshoot arc has one point on S_j that maps to $[a_j, b_j]$, there is an offshoot endpoint p_{j+1} of level at least $k + 1 - j$ that maps into (a_j, b_j) that is close enough to p_j to not be in any S_i, $i < j$; it misses S_j by definition of offshoot limit point.

Thus the process can continue inductively until $[a_{k+1}, b_{k+1}]$ has too many point inverses for each of its points. Q.E.D. (for Fact 1)

Fact 2. Suppose F' is a closed set in X' containing F and suppose P is a closed set in $X' - F'$ with no offshoot limit points. Then there is a finite collection of arcs in X' whose union contains P such that each component of this union intersects F'.

PROOF. Well-order P. Let S_1 be an arc from the first point p_1 of P to F'. If p_a and S_a have been selected for each $a < b$, let p_b be the first point of P not in $B_b = \mathrm{Cl}(\bigcup S_a | a < b)$, and let S_b be an irreducible arc from p_b to B_b. Finally, let c be the least ordinal greater than each ordinal used, and let $B_c = \mathrm{Cl}(\bigcup S_a | a < c)$.

Claim. (1) Each B_b is the union of finitely many arcs, each component of which intersects F', and (2) if $a < b$ then there is a finite set of points in $P, p_1, p_2, \ldots, p_n$, such that B_b is B_a plus an irreducible arc T_1 from p_1 to $F' \cup B_a$, plus an irreducible arc T_2 from p_2 to $F' \cup B_a \cup T_1, \ldots,$ plus an irreducible arc T_n from P_n to $F' \cup B_a \cup T_1 \cup \cdots \cup T_{n-1}$. Furthermore, if b is a limit ordinal then there are on each T_i points of P arbitrarily close to p_i.

Let b be the least ordinal such that the claim is false. The first, $B_2 = S_1$, satisfies part (1) by construction and part (2) vacuously. If b has a predecessor, then B_b is B_{b-1} plus a new arc S_{b-1} irreducible from p_{b-1} to $F' \cup B_{b-1}$, and so satisfies part (2) by induction and part (1) follows from part (2).

Now suppose b is a limit ordinal. First, b cannot be an uncountable ordinal since each $B_{a+1} - B_a$ contains an arc. If b is uncountable then there is an uncountable collection of disjoint arcs and an infinite subcollection whose diameters are bound away from 0 (contradicting Lemma 0). Since b is countable, then, there is a sequence of increasing ordinals, $b1, b2, \ldots$ whose limit is b. Let $b1 = 2$, the first relevant ordinal.

Consider the structure of these B_{bi}. The first, B_2, is an arc from a point of P to F'. Then, for B_{b2}, there is a finite subset of P satisfying part (2) of the claim for $b2$ and $b1 = 2$. Consider one of the arcs T_j irreducible from p_j to $F \cup B_{b1} \cup T_1 \cup \cdots \cup T_{j-1}$. The arc T_j either goes directly to F' or to one of the previous T_i, or to S_1.

In the process of building the B's only finitely many T's can go to F'. Since the T's are disjoint (without their endpoints), any infinite collection is null. So if infinitely many T's intersected F' then some point of F' would be a limit of T's and hence of P since each T has a point of P. This contradicts the hypothesis that F' and P are disjoint closed sets.

Back to the more prevalent case then: suppose T_j intersects a previous T_i and its non-p_j endpoint is p_i, the endpoint of T_i. If T_j is the first T past T_i to do this then T_j merely extends the arc T_i and no triod is formed. Otherwise, if T_j hits T_i at an interior point, or if some previous T extended T_i and T_j hits T_i at p_i, then a triod is formed whose three endpoints are all in P. Another way that a triod can be formed is if some $p_i = p_j$ at a limit level, or if the limit p_i belongs to some previous T. Such a point p_i is a possible junction point and any triod with p_i as junction point either has regular T legs or arbitrarily short legs tipped with points from P from the "furthermore" part of the claim. We will show that there are only finitely many triods formed in this construction, namely:

Subclaim. There are only finitely many B_{bi} that contain triods.

Suppose a single T_3 from some B_{bj} is abutted by infinitely many later T's from B_{bj} and later B's in the sequence. As before, the T's form a null sequence, and a convergent subsequence converges to some point of T_3 which is an offshoot limit point of P since each T has a point of P. But P has no offshoot limit points.

Now suppose that there are infinitely many triods formed using all different T's. The triods form a null sequence and some subsequence $t_1, t_2, \ldots$ converges to the point p_0 in P. Since X is locally connected, there is a short arc C_1 from p_0 to t_1, a shorter arc C_2 from p_0 to t_2, etc., so that $\{C_i\}$ converges to p_0, and none intersect F'.

For each $i > 1$, there is an arc in $t_i \cup C_i$ from each endpoint of t_i to the first arc C_1. If p_0 is not to be an offshoot limit point of P, all except finitely many of the endpoints of the triods must be in C_1. We will suppose they all are and we will assume that p_0 does not belong to any of the triods, since p_0 only belongs to at most one. The endpoints of t_2 are in C_1; replace the subarc of C_1, between the first point of C_1 in t_2 and the last point of C_1 in t_2, with the subarc of t_2 between the same two points. Since no arc in t_2 contains all three endpoints, one of them is no longer in C_1. Closer to p_0 there is another triple of endpoints whose triod misses t_2 and another similar substitution can be made. After doing this for infinitely many triods, the new arc with its offshoot triod arcs to P, makes p_0 an offshoot limit point of P; a contradiction. This establishes the subclaim.

Thus, for some i, every new arc that goes to B_{bi} simply extends some previous arc. If B_{bi} has, say, n endpoints, then so does each B_{bj} for $bj > bi$. Each ray produced by tacking one arc to the end of the previous arc has a unique limit point, since the sequence of arcs is null. These n rays, plus their limit points, form n arcs that decompose $B_b - B_{bi}$, and there are, as required in the "furthermore" part of the claim (2), points of P arbitrarily close to the limit points of the rays. If $a < b$, then some bj past bi is greater than a and by induction, $B_{bj} - B_a$ is a successive arc buildup as required, and by using final segments of the rays (plus their limit points) to structure $B_b - B_{bj}$, the difference $B_b - B_a$ has the structure required to satisfy the rest of part (2) of the claim. Part (1) follows from the fact that B_b is the union of B_{bi} and the n arcs of $B_b - B_{bi}$.

This establishes the claim, completes the induction, and proves Fact 2.

Note. If $P \subset U$, an open set in X' with the property that for each point x in P there is an arc from x to F' in U, then each arc in the finite collection that satisfies the conclusion of Fact 2 is in U if the original arcs, S_a, are constructed in U in the beginning.

(Proof of the main claim, continued.) Recall that the aim is to decompose $X' - F$ into a collection of disjoint open or half open, half closed arcs.

Let $\{e_i\}$ denote a sequence of positive numbers converging to 0 with the property that if q belongs to the e_i-neighborhood of F, denoted $N_{ei}(F)$, there is an arc from q to F in $N_{e,i-1}(F)$. This sequence exists since F is compact and X' is locally connected.

For each i, let $P(i)$ denote the i-level (of complexity) points in $X' - F$. Define $P_{k0} = P(k) - N_{e1}(F)$ and for $i > 0$, define $P_{ki} = P(k) \cap (\mathrm{Cl}(N_{ei}(F) - N_{e,i+1}(F)))$. Since there are no $(k+1)$-level points outside of F (Fact 1), each P_{ki} is closed, misses F, and has no offshoot limit points. Hence from Fact 2, there is for each i, a finite collection H_i of arcs such that $P_{ki} \subset \bigcup H_i$, and each component of the union of the elements of H_i has a point of F. If $i > 1$, make $\bigcup H_i \subset N_{e,i-1}(F)$. (See the note at the end of the proof of Fact 2.)

Let S_1 be an arc in H_0 containing a point of F. Each component of $S_1 - F$ is an open or half open, half closed arc. Put in the collection L_{k0} (being constructed) each component of $S_1 - F$ that contains a point of $P(k)$.

Each component of $\bigcup H_0$ intersects F so there is a second arc S_2 in H_0 (if $H_0 \neq \{S_1\}$) that intersects S_1 or F. Add to L_{k0} the open or half open, half closed arc components of $S_2 - (S_1 \cup F)$ that contain points of $P(k)$. For each $e > 0$, there are only finitely many L_{k0} members outside $N_e(F)$ since no point outside of F is an offshoot limit point of $P(k)$. Continue this process with the other arcs of H_0 one at a time. So far:

(1) $\bigcup L_{k0}$ contains P_{k0},

(2) $\mathrm{Cl}(\bigcup L_{k0}) - \bigcup L_{k0} \subset F$,

(3) the members of L_{k0} are disjoint open or half open, half closed arcs that miss F, only finitely many of which are more than e away from F, for any $e > 0$, and

(4) $\bigcup L_{k0} \subset \bigcup H_0$.

Now for H_1. Some arc T_1 in H_1 intersects F. Each component of $T_1 - (F \cup (\bigcup L_{k0}))$ is an open or half open, half closed arc since $F \cup (\bigcup L_{k0})$ is closed; those components that contain a point of $P(k)$ will be put in L_{k1}. Continue this process on the other members of H_1, one at a time. Now:

(i) $(\bigcup L_{k0}) \cup (\bigcup L_{k1})$ contains $P_{k0} \cup P_{k1}$,

(ii) the previous properties (2) and (3) hold with L_{k0} replaced by $L_{k0} \cup L_{k1}$, and

(iii) $\bigcup L_{k1} \subset (\bigcup H_1) \subset N_{e0}(F)$.

Continue and let $L_k = \bigcup\{L_{ki} | i = 0, 1, \dots\}$. From (i), $P(k) \subset \bigcup L_k$. For each $e > 0$, there is an i such that $\bigcup\{L_{kj} | j > i\} \subset N_e(F)$, and each of $L_{k0}, L_{k1}, \dots, L_{ki}$ has only finitely many members outside of $N_e(F)$. This fact together with property (2) and the fact that any infinite sequence of the disjoint L_k is null, ensures that $\mathrm{Cl}(\bigcup L_k) - \bigcup L_k \subset F$. The earlier L_{ki} are not changed so property (3) ensures that L_k is a collection of disjoint open and half open, half closed arcs that miss F.

For $P(k-1)$, use in the place of F the closed set $F' = F \cup (\bigcup L_k)$. Then F' contains all offshoot limit points of $P(k-1)$ since they are all in $F \cup P(k)$. The same construction yields L_{k-1}, a collection of disjoint open or half open, half closed arcs that miss F' and whose union contains those points of $P(k-1)$ not already in $\bigcup L_k$.

Finally, $L_k \cup L_{k-1} \cup \cdots \cup L_0$ will decompose all of $X' - F$ into disjoint open or half open, half closed arcs. Q.E.D. (Main claim)

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